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RESEARCH ARTICLE

ABSTRACT FOR SYMPLECTIC YANG-MILLS FIELDS

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ABSTRACT

In the realm of Abstract Differential Geometry (à la Mallios) according to Abstract Symplectic Geometry, we study symplectic Yang-Mills fields. We generalize the classical result established by K. Habermann, L. Habermann and P. Rosenthal about the variational principle to symplectic A-connections on a symplectic vector sheaf (E,σ). Special attention is given to the set of symplectic A-connections, $Conn_A(E,\sigma)$, and to the moduli space of the symplectic Yang-Mills fields.

Key words: Vector sheaves, Yang-Mills field, Symplectic A-connection, A-symplectomorphism, Moduli space of symplectic A-connections.

INTRODUCTION

In Classical Differential Geometry (CDG) of $\mathbb{C}^\infty$-manifolds, symplectic gauge theories are built on principal bundles and their connections, see for instance (Mitter and Viallet, 1981), (Eguchi *et al*., 1980), (Jost, 2008). The aim of this paper is to develop the abstract symplectic gauge theories without any differentiability by enlarging the smooth $\mathbb{C}^\infty$-structures to A-structures with A ≡ (A, τ, X), a sheaf of commutative, associative and unital ℂ-algebras over a topological space X. The structure of symplectic classical theories survives in the Abstract Differential Geometry (ADG). In (Habermann *et al*., 2006), Habermann *et al*. generalize the variational principle for symplectic connections developed in (Bourgeois and Cahen, 1999) by F.Bourgeois and M.Cahen to connections on vector bundles. Based on the work in (Habermann *et al*., 2006), we built the variational principle for symplectic A-connections on a symplectic vector sheaf. The starting point of our study is the action of SpE, the group sheaf of symplectomorphisms of a symplectic vector sheaf (E,σ) on the set of all symplectic A-connections on (E,σ). Adapting abstract Laplace-Beltrami operator suggested by A.Mallios in (Mallios, 2010) to symplectic vector sheaves, we set up the symplectic Yang-Mills equations.

Symplectic vector sheaves

Definition 2.1 A vector sheaf E on a given topological space X is a locally free A-module of finite rank n over X, i.e for any open subset U of X

$$E|_U = A^n|_U = (A|_U)^n. \qquad (1)$$

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Definition 2.2 Let E be a vector sheaf on a given topological space X. A sheaf morphism $\sigma : E \oplus E \to A$ which is A-bilinear, skew-symmetric, and nondegenerate is called a symplectic A-form on E. (Anyaegbunam, 2010)

If $E^* = Hom_A(E,A)$ is the dual vector sheaf of E, the map $\tilde{\sigma} : E \to E^*$ defined by $\tilde{\sigma}(s)(t) = \sigma(s,t)$ for any sections s, t ∈ E(U) and open subset U of X, is an A-isomorphism of vector sheaves.

A given vector sheaf E of even rank on a topological space X equipped with a symplectic A-form σ is said to be a symplectic vector sheaf on X which is denoted by (E,σ).

Definition 2.3 Let E be a symplectic vector sheaf on a given topological space X. A morphism sheaf $\varphi : E \to E$ such that $\sigma \circ (\varphi, \varphi) = \sigma$ is said to be an A-symplectomorphism of E.

For any sections s, t ∈ E(U),

$$\sigma \circ (\varphi, \varphi)(s,t) = \sigma(\varphi(s),\varphi(t)) = \sigma(s,t) \qquad (2)$$

with U an open subset of X.

Definition 2.4 The set

$$SpE = \{\varphi \in AutE : \sigma \circ (\varphi, \varphi) = \sigma\} \qquad (3)$$

is a subgroup of AutE called the group sheaf of symplectomorphisms of E.

According to the fact that E is a vector sheaf of rank 2n, $E|_U = A^{2n}|_U = (A|_U)^{2n}$, for any open subset U of X,

$$SpE(U) = (SpE|_U)(U) = Sp(A^{2n})(U). \qquad (4)$$

Symplectic a-connections

Definition 3.1 Let E be an A-module on a topological space X

and $\partial : A \rightarrow E$ a sheaf morphism. The triplet (A,∂,E) constitutes a differential triad, if it satisfies the following conditions :
(i) ∂ is a $\mathbb{C}$-linear morphism,

(ii) for any $s,t \in A(U)$, $\partial(s.t) = \partial(s).t + s.\partial(t)$. (5)

Definition 3.2 Let (A,∂,Ω) be a differential triad on a topological space X and E an A-module on X.A sheaf morphism

$$\nabla \in Hom_{\mathbb{C}}(E, E\otimes\Omega) \quad (6)$$

such that for any $\alpha \in A(U)$ and $s \in E(U)$,

$$\nabla(\alpha s) = \alpha\nabla(s) + \partial(s) \otimes \alpha \quad (7)$$

is called an A-connection of the vector sheaf E.

Let U be an open subset of X, $\alpha \in A(U)$ and $s \in E(U)$,

$$\nabla(\alpha s) = \alpha\nabla(s) + s \otimes \partial\alpha. \quad (8)$$

In particular, if E is a vector sheaf on X the pair (E,∇) is called a Yang-Mills field, (see (Mallios, 2005)).

Proposition 3.3 Let E be a vector sheaf on a topological space X and Ω be an A-module of 1-forms, then

$$E \otimes_A \Omega = End_A(E) \quad (9)$$

within an A-isomorphism.

Proof. From $\Omega = Hom_A(E, A)$, one can write
$E \otimes_A\Omega = E \otimes_A Hom_A(E,A)$
$E \otimes_A\Omega = Hom_A(E,E) \otimes_A A$
$E \otimes_A\Omega = End_A(E)$.

Using the result of the proposition 3.3 and the definition 3.2, one considers ∇ as the sheaf morphism $\nabla : E \rightarrow End_A(E)$.

This implies $\nabla \in Hom_{\mathbb{C}}(E, End_A(E))$.
For any open subset U of X, s and $t \in E(U)$,
$\nabla(s) \in End_{A(U)}(E(U))$ and $\nabla(s)(t) \in E(U)$.
We notice that ∂ can be seen as an A-connection of A in the sense that $\partial \in Hom_{\mathbb{C}}(A,\Omega) = Hom_{\mathbb{C}}(A,A\otimes\Omega)$.
For any open subset U of X, $\alpha \in A(U)$ ans $s \in A(U)$, one has $\partial(U) \in \Omega(U) = Hom_{A(U)}(E(U),E(U))$ and
$\partial(\alpha)(s) \in A(U)$.

The set of all A-connections on E is an affine space denoted by $Conn_A(E)$.

Definition 3.4 Let (E,σ) be a symplectic vector sheaf on a topological space X. ∇ an A-connection on E is said to be a symplectic A-connection on E if $\nabla\sigma = 0$ i.e for any U open subset of X and sections $s,t,r \in E(U)$, one has :

$$\partial(\sigma(t,r))(s) = \sigma((s)(t),r) + \sigma(t,(s)(r)). \quad (10)$$

In (Habermann *et al.*, 2006), the authors give the definition of symplectic connection for the classical case.

Proposition 3.5 Let E be a vector sheaf on a topological space X and Ω be an A-module of 1-forms, then

$$\Omega(End_AE) = Hom_A(E,End_AE) \quad (11)$$

within an A-isomorphism.

Proof. Since $End_AE = E\otimes_A\Omega$ from the proposition 3.3, it follows that

$\Omega(End_AE) = \Omega(E\otimes_A\Omega) = (E\otimes_A\Omega)\otimes_A\Omega = End_AE\otimes_A\Omega$
$\Omega(End_AE) = Hom_A(E,E)\otimes_A\Omega = Hom_A(E,E\otimes_A\Omega)$
$= Hom_A(E,End_AE)$.

We denote by $Conn_A(E,\sigma)$ the set of all symplectic A-connections on (E,σ).

The A-module

$$\Omega(End_A(E,\sigma)) = Hom_A((E,\sigma), End_A(E,\sigma)) \quad (12)$$

is the sub-A-module of $\Omega(End_AE)$ such that
$u \in \Omega(End_A(E,\sigma))$ iff $u \in \Omega(End_AE)$ and
$\sigma(u(s)(t),r)+\sigma(t,u(s)(r)) = 0$ for any U open subset of X and sections $s,t,r \in E(U)$. For a fixed symplectic A-connection ∇ on (E,σ), the map defined by $u \rightarrow \nabla + u$ is a bijection,
$Hom_A((E,\sigma), End_A(E,\sigma)) \approx Conn_A(E,\sigma)$. (13)
Thus, one can find one symplectic A-connection ∇' on (E,σ) such that $\nabla' = \nabla + u$ and $\nabla' - \nabla \in \Omega(End_A(E,\sigma))$

Proposition 3.6 Let (E,σ) be a symplectic vector sheaf on a topological space X, $Conn_A(E,\sigma)$ is an affine space modeled on $\Omega(End_A(E,\sigma))$.

Proof. Consider the map

$\psi : Conn_A(E,\sigma) \times Conn_A(E,\sigma) \rightarrow \Omega(End_A(E,\sigma))$
defined for any $\nabla, \nabla' \in Conn_A(E,\sigma)$ by $\psi(\nabla,\nabla') = \nabla' - \nabla$.
This map satisfies the two following conditions:
(i) for any $\nabla, \nabla', \nabla'' \in Conn_A(E, \sigma)$
$\psi(\nabla,\nabla') + \psi(\nabla',\nabla'') = (\nabla' - \nabla) + (\nabla'' - \nabla') = \nabla'' - \nabla$
$= \psi(\nabla,\nabla'')$.
(ii) for any $\nabla \in Conn_A(E,\sigma)$, $u \in \Omega(End_A(E,\sigma))$,
$\nabla + u \in Conn_A(E, \sigma)$ thus there exists $\nabla' = \nabla + u$ belongs to $Conn_A(E, \sigma)$ such that $\nabla' - \nabla = u = \psi(\nabla,\nabla')$.
For a given symplectic vector sheaf (E,σ) and ∇ a symplectic A-connection on (E,σ),
$Conn_A(E,\sigma) = \nabla + \Omega(End_A(E,\sigma))$. (14)

Moduli space of symplectic a-connec-tions

Definition 4.1 Let (E,σ) be a symplectic vector sheaf on a given topological space X and ∇ a symplectic A-connection on (E,σ), the pair (E,∇) is called a symplectic Yang-Mills field.

Given a symplectic vector sheaf (E,σ) on a given topological space X, two symplectic A-connections ∇ and ∇' on (E,σ) are related if there exists an A-symplectomorphism $\varphi \in SpE$ such that

$$\nabla' \circ \varphi = (\varphi\otimes 1_\Omega) \circ \nabla. \quad (15)$$

Using the fact that $\varphi \in SpE$ is an A-isomorphism,

$\varphi^{-1} \in SpE$, one gets $\nabla' = (\varphi\otimes 1_\Omega) \circ \nabla \circ \varphi^{-1} = \varphi\nabla\varphi^{-1}$.
The group sheaf of symplectomorphisms of E acts on the set of all symplectic A-connections as follows:

$$SpE \times Conn_A(E,\sigma) \rightarrow Conn_A(E,\sigma) ; (\varphi,\nabla) \rightarrow \varphi\nabla\varphi^{-1}. \quad (16)$$

Proposition 4.2 The action of the group sheaf of symplectomorphisms of E on $Conn_A(E,\sigma)$ defines an equivalence relation on $Conn_A(E,\sigma)$, $\nabla\sim\nabla'$ if and only if there exists an A-symplectomorphism $\varphi\in SpE$ such that $\nabla'=\varphi\nabla\varphi^{-1}$.

Proof. (i)For any $\nabla\in Conn_A(E,\sigma)$, according to the fact that $1_E\in SpE$ it is obvious that $\nabla\sim\nabla$.

(ii)For any $\nabla,\nabla'\in Conn_A(E,\sigma)$, if $\nabla\sim\nabla'$, there exists $\varphi\in SpE$ such that $\nabla'=\varphi\nabla\varphi^{-1}$. As φ is A-isomorphism, $\varphi^{-1}\in SpE$ and $\varphi^{-1}\nabla'\varphi=\varphi^{-1}(\varphi\nabla\varphi^{-1})\varphi=(\varphi^{-1}\varphi)\nabla(\varphi^{-1}\varphi)=\nabla$, one concludes that $\nabla'\sim\nabla$.
(iii) For any $\nabla,\nabla',\nabla''\in Conn_A(E,\sigma)$, if $\nabla\sim\nabla'$ and $\nabla'\sim\nabla''$ there exists $\varphi\in SpE$ such that $\nabla'=\varphi\nabla\varphi^{-1}$ and $\psi\in SpE$ such that $\nabla''=\psi\nabla'\psi^{-1}$ then $\nabla''=\psi(\varphi\nabla\varphi^{-1})\psi^{-1}=(\psi\varphi)\nabla(\varphi^{-1}\psi^{-1})$,
$\nabla''=(\psi\varphi)\nabla(\psi\varphi)^{-1}$. Thus $\nabla\sim\nabla''$.

We denote by (∇) the equivalence class of the symplectic A-connection ∇, $(\nabla)=\{\nabla':\nabla'=\varphi\nabla\varphi^{-1},\varphi\in SpE\}$.

The quotient $Conn_A(E,\sigma)/SpE$ is the set of equivalence classes of the symplectic A-connections on (E,σ).
Note that the equivalence class (∇) is the orbit of ∇.

Definition 4.3 The quotient $Conn_A(E,\sigma)/SpE$ is called the orbit space of the symplectic A-connections on (E,σ) or the moduli space of the symplectic Yang-Mills field (E,∇).

By referring to the principal fiber bundles, (Daniel and Viallet, 1980), (Jost, 2008) also to the principal sheaves (14, p.100-101), (12), the repre-sentation

$$\rho: SpE\rightarrow Aut(Conn_A(E,\sigma)) \quad (17)$$

such that for any $\varphi\in SpE$,
$\rho(\varphi): Conn_A(E,\sigma)\rightarrow Conn_A(E,\sigma)$ allows us to describe the orbit space of ∇ as follows $(\nabla)=\{\rho(\varphi)(\nabla)\,/\varphi\in SpE\}$ and to get the equivalence $\nabla'=\rho(\varphi)(\nabla)=\varphi\nabla\varphi^{-1}\Longleftrightarrow\nabla\sim\nabla'$.

Proposition 4.4 Let $\nabla\in Conn_A(E,\sigma)$, it induces an A-connection on the group sheaf of symplectomorphisms of E.

Proof. For two given vector sheaves E and F on a topological space X, one defines an A-connection on the vector sheaf $Hom_A(E, F)$ by

$$\nabla_{HomA(E,F)}\varphi=\nabla_F\circ\varphi-(\varphi\otimes 1_\Omega)\circ\nabla_E \quad (18)$$

see (Mallios, 1998). Replacing in (18), the vector sheaf $Hom_A(E,F)$ by the group sheaf of symplectomorphisms of E, one obtains

$$\nabla_{SpE}\varphi=\nabla\circ\varphi-(\varphi\otimes 1_\Omega)\circ\nabla \quad (19)$$

with $\varphi\in SpE$.

Given a symplectic Yang-Mills field (E,∇), the A-con-nection on SpE leads to the pair

$$(SpE,\nabla_{SpE}) \quad (20)$$

which is also a Yang-Mills field.

Curvature of a symplectic a-connec-tion

Definition 5.1 Let (E,∇) be a symplectic Yang-Mills fiels on a topological space X, one defines the curvature of the symplectic A-connection by

$$R(\nabla)=\nabla^1\circ\nabla \quad (21)$$

where

$$\nabla^1:\Omega(E)=E\otimes\Omega\rightarrow\Omega^2(E)=E\otimes\Omega^2 \quad (22)$$

is the first prolongation of ∇.

Proposition 5.2 Let E be a vector sheaf on a topological space X, then

$$Hom_A(E,\Omega^2(E))=\Omega^2(End_AE). \quad (23)$$

Proof.
$$\begin{aligned}Hom_A(E,\Omega^2(E))&=Hom_A(E,E\otimes\Omega^2)\\&=Hom_A(E,E)\otimes\Omega^2\\&=End_A E\otimes\Omega^2\\&=\Omega^2(End_AE).\end{aligned}$$

Hence, we remark that $R(\nabla)\in\Omega^2(End_AE)$.

Proposition 5.3 Let ∇ be a symplectic A-connection on E and ∇' belongs to the orbit of ∇, then $R(\nabla')=\varphi\circ R(\nabla)\circ\varphi^{-1}$ with $\varphi\in SpE$.

Proof. Given two symplectic A-connections ∇ and ∇' on a symplectic vector sheaf (E,σ) such that

$$\nabla'\circ\varphi=(\varphi\otimes 1_\Omega)\circ\nabla=\varphi\circ\nabla$$

with $\varphi\in SpE$ is an A-symplectomorphism on E one gets

$$\nabla'=\varphi\circ\nabla\circ\varphi^{-1} \quad (24)$$

It also stands that $\nabla'^1\circ(\varphi\otimes 1_{\Omega^2})=(\varphi\otimes 1_{\Omega^2})\circ\nabla^1$ and for sim-plicity, one writes $\nabla'^1\circ\varphi=\varphi\circ\nabla^1$ or

$$\nabla'^1=\varphi\circ\nabla^1\circ\varphi^{-1}. \quad (25)$$

Since the curvature of ∇', $R(\nabla')=\nabla'^1\circ\nabla'$, using (24) and (25), one establishes

$$\begin{aligned}R(\nabla')&=\nabla'^1\circ\nabla',\\&=(\varphi\circ\nabla^1\circ\varphi^{-1})\circ(\varphi\circ\nabla\circ\varphi^{-1})\\&=\varphi\circ(\nabla^1\circ\nabla)\circ\varphi^{-1}\\&=\varphi\circ R(\nabla)\circ\varphi^{-1}.\end{aligned}$$

Yang-mills functional for symplec-tic vector sheaf

Definition 6.1 Let E be a vector sheaf on a topological space X, $J\in End_AE$ so that $J^2=-id_E$ is called an A-complex structure on E.

Consider (E,∇) a symplectic vector sheaf on X of rank 2n. For a given local gauge $e^U=\{U; e_1, e_2, ..., e_{2n}\}$ of E,
$J\in End_AE$ such that $J_U(e_i)=e_{n+i}$, for any i=1,...,n, is an A-complex structure on E with U a open subset of X.
From (4, p.8), we consider the A-pairing
$\sigma:\Omega^2(End_AE)\oplus\Omega^2(End_AE)\rightarrow A$ defined by

$\sigma(\phi\otimes t\wedge r,\phi'\otimes t'\wedge r') =$
$= \sum(\sigma(\phi\otimes t\wedge r(e_{i1},e_{i2}),\phi\otimes t\wedge r(Je_{i1},Je_{i2})))$ (26)

for any ϕ, $\phi \in End_A E$ and $t,t',r,r' \in \Omega$.

Definition 6.2 Let (E,σ) be a symplectic vector sheaf on a symplectic space (X,ω), the functional

$$SYM : Conn_A(E,\sigma) \to A \quad (27)$$

such that for any $\nabla \in Conn_A(E,\sigma)$,

$$SYM(\nabla)=\frac{1}{2}\int_X \sigma(R(\nabla),R(\nabla))\omega^n/n! \quad (28)$$

is said to be the symplectic Yang-Mills functional of E where R(∇) is a curvature of the symplectic A-connection and $\omega^n/n!$ the symplectic volume form.

In the classical case, the terminology Yang-Mills action or Yang-Mills Lagragian are usually used.

Proposition 6.3 Let (E,σ) be a symplectic vector sheaf on a symplectic space (X,ω). The symplectic Yang-Mills functional is invariant relative to the group sheaf SpE.

Proof. Let $\varphi \in SpE$ and $\nabla \in Conn_A(E,\sigma)$, one gets the symplectic A-connections expressed by $\nabla' = \varphi\circ\nabla\circ\varphi^{-1}$. From the definition of the symplectic Yang-Mills functional $SYM(\nabla') = \frac{1}{2}\int_X\sigma(R(\nabla'),R(\nabla'))\omega^n/n!$, the gauge invariancy of the curvature (see proposition 5.3, p.11), i.e $R(\nabla') = R(\nabla)$, implies that $SYM(\nabla') = \frac{1}{2}\int_X\sigma(R(\nabla),R(\nabla))\omega^n/n! = SYM(\nabla)$.

Definition 6.4 Let $Conn_A(E,\sigma)$ be the set of symplectic A-connections on (E,σ) and $SYM(\nabla)=\frac{1}{2}\int_X\sigma(R(\nabla),R(\nabla))\omega^n/n!$ the symplectic Yang-Mills functional of (E,σ), $\nabla\in Conn_A(E,\sigma)$ which is a stationary point of the functional SYM is called a symplectic Yang-Mills A-connection and this curvature R(∇) is named curvature of Yang-Mills.

Yang-mills equations for symplectic yang-mills fields

Given (X, A) a ℂ-algebraized space endowed with a differential triad (A, ∂, Ω) the n^{th}-prolongation of $\partial : A \to \Omega$, denoted dn is defined from Ω^n to Ω^{n+1} by

$$d^{p+q}(s \wedge t) = d^p(s) \wedge t + (-1)^p s \wedge d^q(t) \quad (29)$$

for any $s \in \Omega^p(U)$, $t \in \Omega^q(U)$, with p, q ∈ IN, see (Mallios, 2010).

Let (E,∇) be a Yang-Mills field, the n^{th}-prolongation ∇^n of the A-connection ∇ is defined by $\nabla^n: \Omega^n(E) \to \Omega^{n+1}(E)$,

$$\nabla^n(s\otimes t) = s\otimes d^n(t) + (-1)^n t\otimes\nabla(s) \quad (30)$$

for any $s \in E$, $t \in \Omega^n(U)$, with n ∈ IN.

Definition 7.1 Let (X, A) a ℂ-algebraized space endowed with a differential triad (A, ∂, Ω), (E,σ) a symplectic vector sheaf on X and a symplectic A - connection on X. The differential operator $\delta^{n+1}: \Omega^{n+1}(E) \to \Omega^n(E)$ defined as follow

$$\sigma(\nabla^n(s), t) = \sigma(s, \delta^{n+1}(t)),\ n \in IN, \quad (31)$$

for any $s \in \Omega^n(E(U))$, $t \in \Omega^{n+1}(E(U))$ with U a open subset of X, is called the dual differential operator of ∇^n.

Definition 7.2 Let (E,∇) be a symplectic Yang-Mills field, the operator $\Delta^n : \Omega^n(E) \to \Omega^n(E)$ defined by

$$\Delta^n = \delta^{n+1} \circ \nabla^n + \nabla^{n-1}\circ \delta^n \quad (32)$$

is said to be the symplectic Laplace-Beltrami operator.

Recall that the proposition (4.4) assume the existence of an A-connection on the group sheaf of symplectomorphisms of a symplectic vector sheaf.

Consider the Yang-Mills field (SpE,∇_{SpE}), let us extend the Laplace-Beltrami operator to SpE,

$$\Delta^n_{SpE}: \Omega^n(SpE) \to \Omega^n(SpE) \quad (33)$$

as follow

$$\Delta^n_{SpE} = \delta^{n+1}_{SpE} \circ \nabla^n_{SpE} + \nabla^{n-1}_{SpE}\,\delta^n_{SpE} \quad (34)$$

where Δ^n_{SpE} is the n^{th}-prolongation of Δ_{SpE} and δ^{n+1}_{SpE} the dual differential operator $\Delta^n_{SpE.}$

For n=2, one gets the following sequence

$$\Omega(SpE) \to \Omega^2(SpE) \to \Omega^3(SpE) \quad (35)$$

and its dual one

$$\Omega^3(SpE) \to \Omega^2(SpE) \to \Omega(SpE) \quad (36)$$

so that

$$\Delta^2_{SpE} : \Omega^2(SpE) \to \Omega^2(SpE) \quad (37)$$

and

$$\Delta^2_{SpE} = \delta^3_{SpE} \circ \nabla^2_{SpE} + \nabla_{SpE}\,\delta^2_{SpE\,.} \quad (38)$$

Definition 7.3 Let (E,∇) be a symplectic Yang-Mills field on X, the two equivalent relations

$$\Delta^2_{SpE}\ (R(\nabla)) = 0 \quad (39)$$

and

$$\delta^2_{SpE}\ (R(\nabla)) = 0 \quad (40)$$

are called the symplectic Yang-Mills equations of (E,∇).

Definition 7.4 Let (E,σ) be a symplectic vector sheaf on a topological space X, a symplectic A-connection ∇ on E such that $\delta^2_{SpE}(R(\nabla)) = 0$ is called a symplectic Yang-Mills A-connection on E.

We notice that the set of symplectic Yang-Mills A-connections on E, denoted $Conn_A(E,\sigma)_{\mathbf{YM}}$, is an affine subspace of $Conn_A(E,\sigma)$.
As in the classical case, definitions 6.4 and 7.4 are equivalent.

Proposition 7.5 Let ∇ be a symplectic Yang-Mills A-connection on (E,σ), then any symplectic A-connection which belongs to the orbit of ∇ is also a symplectic Yang-Mills A-connection on (E,σ).

Proof. Given ∇' in the orbit of ∇, there exists $\varphi \in$ SpE so that $\nabla' = \varphi \circ \nabla \circ \varphi^{-1}$. Since the curvature of a symplectic A-connection is invariant relative to the group sheaf SpE, one gets $\delta^2_{SpE}(R(\nabla')) = \delta^2_{SpE}(R(\nabla)) = 0$.
Thus, $\nabla' \in Conn_A(E, \sigma)_{\mathbf{YM}}$.

For a given symplectic vector sheaf (E,σ), the quotient $Conn_A(E,\sigma)_{\mathbf{YM}}/SpE$ is called the moduli space of the symplectic Yang-Mills A-connections of (E,σ) or the solution space of the symplectic Yang-Mills equations.

Proposition 7.6 The solution space of the symplectic Yang-Mills equations is invariant relative to the group sheaf of symplectomorphims of E.

Proof. In virtue of the propositions 5.3 and 7.5, we obtain the invariancy of $Conn_A(E,\sigma)_{\mathbf{YM}}/SpE$.

Conclusion

We describe the moduli space of the symplectic A-connections on a given symplectic vector sheaf (E,σ) from the left action of the the group sheaf of symplectomorphisms SpE on the set of symplectic A-connections on (E,σ). After developing the symplectic Yang-Mills functional, by using the abstract Laplace-Beltrami operator we establish the symplectic Yang-Mills equations. By analogy with the classical case (smooth case), the space of symplectic Yang-Mills A-connections constitute the set of all solutions of the symplectic Yang-Mills equations which are the stationnary points of the symplectic Yang-Mills functional. The invariance of the curvature of a symplectic A-connection relative to the group sheaf of symplectomorphisms on (E,σ) implies the invariance of the symplectic Yang-Mills functional and the symplectic Yang-Mills solution space under the group sheaf of symplectomorphisms.

REFERENCES

Anyaegbunam, A.C. 2010. Geometric algebra via sheaf theory : A view towards symplectic geometry, PhD, University of Pretoria.

Bourgeois, F., M. Cahen, 1999. A variational principle for symplectic connections, *J. Geom. Phys*., 30,233-265.

Cahen, M., S.Gutt, J. Rawnsley, 2000. Symplectic connections with parallel Ricci tensor, Poisson Geometry, Banach Center Publication,Volume 51*, Institute of Mathematics Polish Academy of sciences*, Warszawa.

Daniel, M., C.M. Viallet, 1980. The geometrical setting of gauge theories of the Yang-Mills type, Reviews of Modern Physics, Vol.52, N.1, American Physical Society.

Eguchi, T., P.B. Gilkey, A.J. Hanson, 1980. Gravitation, Gauge Theories and Differential Geometry, Physics Reports (Review Section of Physics Letters) 66. No.6; 213-393, North-Holland Publishing Company.

Habermann, K., L. Habermann, P. Rosenthal, 2006. Symplectic Yang-Mills theory, Ricci tensor, and connections, arXiv:math/0604553v2 (math.SG).

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