International Journal of Innovation Sciences and Research
Vol.6, No, 06, pp. 1017-1024, June 2017

Available online at
http://www.ijisr.com

RESEARCH ARTICLE

APPLICATION OF CONNECTION TRIADS IN STATIC LIE GROUP AND LIE ALGEBRA

***[1]Chiteng, B. K., [2]Phiri, P. A., [3]Kabwita, P.M., [3]N'gandwe K. Mumba and [4]Tossa, J.**

[1]Department of Mathematics and Informatic, University of Lubumbashi, Lubumbashi, D.R.Congo
[2]Department of Mathematics, Copperbelt University, Kitwe, Zambia
[3]Department of Physics, Copperbelt University, Kitwe, Zambia
[4] Institut de Mathematiques et de Sciences Physiques, Abomey-Calavi, Benin

Accepted 29th May, 2017; Published Online 30th June, 2017

ABSTRACT

In this article we treat differential form triads over the fixed topological space X. Through the Lie group we construct the Lie algebra. We suggest a physic application in the*Poisson Static manifold overX.*

Key words: Differential triads, *Lie group,Lie algebra, Poisson bracket.*

INTRODUCTION

Definition 1.1 *Let us c*onsider the following triplet:

$(Alg_X,\ Diff_X\ ,DMod_X\ )$ ……………………………………[1.1]

such that, for any $\mathcal{A}_{iX} \in Ob(Alg_X)$, there exist $d_{iX} \in Diff_X$ and $\Omega_{iX} \in Ob(DMod_X)$ satisfying, for any open U in X, the Leibniz (product) rule

$d_{iU}(a_i.a'_i) = a_i.d_{iU}(a'_i) + a'_i.d_{iU}(a_i),$ …………………………………… [1.2]

with $a_i, a'_i \in \mathcal{A}_{iU} \equiv \mathcal{A}_i(U)$ where $d_{iU}: \mathcal{A}_{iU} \equiv \mathcal{A}_i(U) \to \Omega_{iU} \equiv \Omega_i(U)$, is continuous and $\mathbb{K}_U$-linear. We set dT_X as a differential triad over $(X, \mathcal{A}_X)$.

$dT_X\ = (\mathcal{A}_X,\ d_X\ ,\Omega_X)$ ……………………………………[1.3]

If $F_X^d: Alg_X \to DMod_X$ is a functor defined, for any $\mathcal{A}_{iX}, \mathcal{A}_{jX} \in Ob(Alg_X)$ and $h_{\mathcal{A}_X}^{ij} \in H_{\mathcal{A}_X}^{ij}$as follows *:*

$F_X^d \big|_{\mathcal{A}_{iX}} := d_{iX}\ ,\ F_X^d \big|_{H_{\mathcal{A}_X}^{ij}} := d_X^{ij}\ ,$ ……………………………………[1.4]

where $H_{\mathcal{A}_X}^{ij} = Hom_{Alg_X}(\mathcal{A}_{iX}\ ,\ \mathcal{A}_{jX})$ and, $d_X^{ij}: H_{\mathcal{A}_X}^{ij} \to H_{\Omega_X}^{ij}$is a continuous map with $H_{\Omega_X}^{ij} = Hom_{DMod_X}(\Omega_{iX}\ ,\ \Omega_{jX})$. The symbol " $|$ " designs the restriction, and in this case the triplets:

$(\mathcal{A}_{iX}, d_{iX}\ ,\Omega_{iX}\ )\ ,\ \ (H_{\mathcal{A}_X}^{ij}\ ,\ d_X^{ij}\ ,H_{\Omega_X}^{ij})$ ……………………………………[1.5]

are differential triads in *Ob* $(Alg_X \times F_X^d \times DMod_X)$ and *Mor* $(Alg_X \times F_X^d \times DMod_X)$*,* respectively; i.e., which satisfy [2.2]. The functor $F_X^d : Alg_X \to DMod_X$ satisfying [2.4] is a differential triad functor over *X.* Note that the Ω_{iX} are sheaves of (differential*)* $\mathcal{A}_X$ -modules over *X,* the $\mathcal{A}_{iX}$ are sheaves of unital $\mathbb{K}$-*algebras* over *X,* d_{iX} and d_X^{ij} are derivative maps as the $\mathbb{K}_X$ -sheaf morphisms which are also $\mathbb{K}_X$-*linears,* where $\mathbb{K}_X = (\mathbb{R}_X \text{ or } \mathbb{C}_X)$.

$dT_{iX}\ = (\mathcal{A}_{iX}\ ,\ d_{iX}\ ,\ \Omega_{iX}), dT_X^{ij} = (H_{\mathcal{A}_X}^{ij}\ ,\ d_X^{ij}\ ,H_{\Omega_X}^{ij})$ …………………………………… [1.6]

Corresponding author: Chiteng, B. K.,
Department of Mathematics and Informatic, University of Lubumbashi, Lubumbashi, D. R. Congo

Definition 1.2 Let dT_{iX} and dT_{jX} be two differential triads over X. The morphism of differential triads between dT_{iX} and dT_{jX}(or simply from dT_{iX} *to* dT_{jX}) is the triplet

$$(h^{ij}_{\mathcal{A}_X},\ d^{ij}_X,\ h^{ij}_{\Omega_X}), \qquad \text{.....[1.7]}$$

where $h^{ij}_{\mathcal{A}_X} \epsilon\, H^{ij}_{\mathcal{A}_X})$ and $h^{ij}_{\Omega_X} \epsilon\, H^{ij}_{\Omega_X}$ are continuous maps and d^{ij}_X satisfies, for any open U in X, the relation:

$$d^{ij}_U\left(h^{ij}_{\mathcal{A}_U}\right) = h^{ij}_{\Omega_U}) \qquad \text{..... [1.8]}$$

and the Leibniz (product) rule:

$$d^{ij}_U(h^{ij}_{\mathcal{A}_U} . h^{ij'}_{\mathcal{A}_U}) = h^{ij}_{\mathcal{A}_U} . d^{ij}_U\left(h^{ij'}_{\mathcal{A}_X}\right) + h^{ij'}_{\mathcal{A}_U} . d^{ij}_U(h^{ij}_{\mathcal{A}_U}), \qquad \text{.....[1.9]}$$

with $h^{ij}_{\mathcal{A}_U}, \ h^{ij'}_{\mathcal{A}_U} \epsilon\, H^{ij}_{\mathcal{A}}(U) \equiv H^{ij}_{\mathcal{A}_U}$. The map $d^{ij}_U : H^{ij}_{\mathcal{A}_U} \to H^{ij}_{\Omega_U}$ is continuous.

We observe that, for any $U \subseteq X$ we have:

$$h^{ij}_{\Omega_X}(a_i \,.\, \omega_i) = h^{ij}_{\mathcal{A}_X}(a_i).\, h^{ij}_{\Omega_X}(\omega_i), \qquad \text{.....[1.10]}$$

where $(a_i, \omega_i)\ \epsilon\ \mathcal{A}_{iU} \times \Omega_{iU}$.

We denote the morphism of differential triads (or simply a differential triad morphism*):*

$$dT_{iX} \text{ and } dT_{jX} \text{ by : } \ mdT_X{}^{ij} = (h^{ij}_{\mathcal{A}_X},\ d^{ij}_X,\ h^{ij}_{\Omega_X}) \qquad \text{.....[1.11]}$$

So that:

$$d^{ij}_X(h^{ij}_{\mathcal{A}_X})\ \Big|_{a_i} = (h^{ij}_{\Omega_X} \circ d_{iX})\ (a_i) = (d_{jX} \circ h^{ij}_{\mathcal{A}_X})(a_i), \qquad \text{.....[1.12]}$$

for any $a_i \epsilon \mathcal{A}_{iU}$, where the symbol " | " designs the restriction.

Theorem 1.3 The composition of morphism of differential triads is associative.

Proof. It is proved in *[14]*.

Definition 1.4 The differential triads dT_{iX} and their morphisms mdT^{ij}_X*; i,j = 1,2,3,...* form the category, denoted $DiffT_X$ and called *the category of differential triads over X.*

Note that we can also generalize the same notions to the category $Open_X$ and construct the *category of differential triads* over $Open_X$ or *TOP* denoted, respectively by:

$DiffT_{Open_X}$, $DiffT_{TOP} \equiv$ *DiffT,* with $DiffT_X \subseteq DiffT_{Open_X} \subseteq$ *DiffT.*

2. Differential Form Triads

Definition 2.1

Now consider, for any $x \in X$, the tangent space T_xX. Hence, we define the sheaf

$$TX := \cup_{x \in X} T_xX \equiv \textstyle\sum_{x \in X} T_xX \qquad \text{.....[2.1]}$$

as the *tangent bundle sheaf.* Referring to the Classical Differential Geometry, in short CDG, it follows that if X is a smooth manifold of dimension n and TX is a smooth manifold of dimension $2n$.

Remark 2.2

Consider the morphism $f : X \to Y$, where X , Y are two smooth manifolds of class C^k, ($k \geq 2$).

We set:

$$X^1 = T(X) \equiv TX, \quad Y^1 = T(Y) \equiv TY \qquad \text{.....[2.2]}$$

Then , the *derivative function* f' of f, is such that $f' \equiv Tf \equiv T_f \in C^k(X^1, Y^1)$.

Definitions 2.3

Let $(T\mathcal{A}, \tau, TX)$ and $(T\Omega, \omega, TX)$ be two sheaves of smooth manifolds. If we consider the map Td: $T\mathcal{A} \to T\,\Omega$, it follows that

$Td \circ \omega = \tau \;, Td_{TU}(T\mathcal{A}(TU)) \subseteq T\Omega(TU)$ ……………………………………..[2.3]

We observe that $(T\mathcal{A}_X, Td_X, T\Omega_X)$ is the *tangent bundle differential triad over* $(T\mathcal{A}, TX)$.

Definitions 2.4

Let us associate to T the funtorial character morphism and we set $T(\mathcal{A}_X, d_X, \Omega_X) = (T\mathcal{A}_X, Td_X, T\Omega_X) \equiv (\mathcal{A}_{TX}, d_{TX}, \Omega_{TX})$.

In this case, T is the *tangent bundle functorial morphism* and one could write:

$T(dT_X) \equiv dT_{TX} := (\mathcal{A}_{TX}, d_{TX}, \Omega_{TX})$ …………………………………[2.4]

We can also call it as the *tangent bundle differential triad* over $(T\mathcal{A}, TX)$.

Remark 2.5

We observe that if $f : X \to Y$ is a morphism of topological space, then we obtain the following commutative diagram (Figure 1)

$$\begin{array}{ccc} & T(d_{iX}) & \\ T(\mathcal{A}_{iX}) & \to & T(\Omega_{iX}) \\ & T(d_f^{ij}) & \\ T(h^{ij}_{\mathcal{A}_f}) \downarrow & \to & \downarrow T(h^{ij}_{\Omega_f}) \\ & T(d_{jY}) & \\ T(\mathcal{A}_{jY}) & \to & T(\Omega_{jY}) \end{array}$$

Figure 1. ***Morphisms of differential tangent bundle triads***

Note that by convenience, we set

$T(d_{iX}) = d_{iTX}$, $T(d_f^{ij}) = d_{Tf}^{ij} \equiv d_{T_f}^{ij}$, $T(d_{jY}) = d_{jTY}$. …………………………………[2.5]

Definition 2.6

Consider the map $s: X \to TX$, $x \to T_xX$ which satisfies the relation

$p_1 \circ s = id_X$, ……………………………….[2.6]

where $p_1 : TX \to X$ *is a projection morphism.* Then, we observe that s is the *vector field* on X.

If $f : X \to Y$ is a morphism of topological space and $\bar{s} : Y \to TY$ is a vector field on Y, it follows that we have :

$T_f = \bar{s} \circ f \circ s^{-1}$ ………………………………[2.7]

where $T_f : TX \to TY$ is a morphism of tangent bundles.

The set of *vector fields of X* will be denoted by $\Xi(X)$ and consequently the sets of *vector fields* of and Ω shall be denoted by $\Xi(\mathcal{A})$ and $\Xi(\Omega)$, respectively.

Remark 2.7

Let $s \in \Xi(X)$ and associate to s a transformation $\varphi : \mathcal{A} \times X \to X$, $(\lambda, x) \to \varphi(\lambda, x)$ such that:

$\frac{d\varphi}{d\lambda} = s$ ……………………………………..[2.8]

where $\lambda \in \mathcal{A}$ *is a parameter.* It follows that for any $\lambda, t \in \mathcal{A}$ we have the relation

$\varphi_\lambda \circ \varphi_t = \varphi_{\lambda+t}$, ……………………………..[2.9] with $\varphi_\lambda : X \to X, x \to \varphi_\lambda(x) = \varphi(\lambda , x)$ and $\varphi_t : X \to X, x \to \varphi_t(x) = \varphi(t , x)$.

Definitions2.8

Let E and F be free $\mathcal{A}$ −modules on X. We denotes by $\mathcal{L}^e_{p\mathcal{A}}(E ,) \equiv Hom^e_{p\mathcal{A}}(E ,) \equiv \wedge^p(E) \equiv \Omega^p(E)$.

The *sheaf of exterior product* form (or of *differential* p *−forms*) as a $\mathbb{K}$ −algebra structure sheaf. Note that we set:

$$\mathcal{L}^e_{p\mathcal{A}}(E , \mathcal{A}) (X) = \mathcal{L}^e_{p\mathcal{A}(X)}(E(X) , \mathcal{A}(X)) , \quad (\mathcal{L}^e_{p\mathcal{A}}(E , \mathcal{A}))_x = \mathcal{L}^e_{p\mathcal{A}_x}(E_x , \mathcal{A}_x) \quad \text{…………………[2.10]}$$

and, for any $x \epsilon U \subseteq X$, we have:

$$(\mathcal{L}^e_{p\mathcal{A}}(E , \mathcal{A}))_x = \varinjlim_{x\epsilon U} (\mathcal{L}^e_{p\mathcal{A}}(E , \mathcal{A}))_U \quad \text{………………[2.11]}$$

Definition 2.9

Let Ω^p be the *sheaf of exterior product forms* of degree p (or of *differential* p *−forms*) as a $\mathbb{K}$ −algebra structure sheaf and consider the morphism $d^p{}_X : \Omega^p{}_X \to \Omega^{p+1}{}_X$. The triplet

$$(\Omega^p{}_X , d^p{}_X , \Omega^{p+1}{}_X) \quad \text{…………………...[2.12]}$$

is a triad of differential p *−forms* relative *to* $(X , \mathcal{A}_X)$ *iff:*

$$\Omega^p{}_X = \mathcal{A}_X, \text{ if } p=0 \quad \text{…………………......[2.13]}$$

and, for any open U in X, the Leibniz (product) rule

$$d^p{}_U(w.w') = w.d^p{}_U(w') + w'.d^p{}_U(w) \quad \text{………………….[2.14]}$$

is satisfied , with w, $w' \epsilon \Omega^p{}_U$ and $d^p{}_U : \Omega^p{}_U \to \Omega^{p+1}{}_U$ is a continuous map. We set: $dT^p{}_X := (\Omega^p{}_X , d^p{}_X , \Omega^{p+1}{}_X)$

Definition 2.10

Let $\mathcal{A}_X = C^\infty_X$ be the structure sheaf of germs of local $\mathbb{R}$ (or $\mathbb{C}$)-valued C^k-functions on X, and $\Omega_X = \Omega_X{}^1$ as the sheaf of germs of its smooth $\mathbb{R}$ (or $\mathbb{C}$)-valued 1-forms then , we obtain

$$dT^\infty{}_X := (C^\infty_X , d_X, \Omega_X{}^1) \quad \text{…………………[2.15]}$$

and say that [2.15] is a *differential triad of smooth manifolds* on X (or simply *a manifold differential triad* of X). *Thus, the concept of a differential triad generalizes that of a manifold.*

If we specify the order of differential forms of Ω ($\mathcal{A}$-sheaf of differential-modules) by setting

$$\Omega^i \equiv (\Omega^1)^i = \wedge\Omega^1 \text{ , with } \wedge \equiv \wedge_{\mathcal{A}} \text{ , i = 1,2,3,…} \quad \text{…………………….[2.16]}$$

With $\Omega^0 :=$, $\Omega^1 := \mathcal{A} \wedge \Omega$, $\Omega^2 := \mathcal{A} \wedge \Omega^1 \wedge \Omega^1$, …; where $\wedge$ is *the skew symmetric homological tensor product* (see[12]) - [14]).

$$d^{-1}{}_X \equiv \varepsilon_X, \quad d^0{}_X \equiv \partial_X, \quad d^1{}_X \equiv d_X, \quad d^2{}_X \equiv d_X, \quad d^p{}_X \equiv d_X$$

$0 \to \mathbb{C}_X \to \mathcal{A}_X \to \Omega_X \to \Omega^2{}_X \to ... \to \Omega^p{}_X \to \Omega^{p+1}{}_X \to …$, is co-homologically exact , i.e., we have :

$$ker\ \Omega^{p+1}{}_X = Im\ \Omega^p{}_X, \quad d^{p+1} \circ d^p \equiv d \circ d = 0 , \quad p = 0,1,2,…$$

3.Lie Algebra Triads

Definitions 3.1

Let X be a fixed smooth manifold. A *Lie group sheaf* over X is define as a sheaf of smooth manifolds endowed with a group sheaf structure, such that the map $\varphi_X : G_X \times G_X \to G_X$ is differentiable and , for any $(g , g') \epsilon G_U \times G_U$ *with* $U \subseteq X$, *we have:*

$$\varphi_U(g, g') = g.g'^{-1} \quad \text{……………………………………..[3.1]}$$

Definitions 3.2

The *Lie algebra sheaf* $\mathfrak{G}\mathfrak{G}_X$ of the Lie group G_X is defined as a vector sheaf isomorphic to a tangent space sheaf T_eG_X, with e $\epsilon\ G_X$; in other words,

$$Dim\ \mathfrak{G}\mathfrak{G}_X = dimT_eG_X = dim\ G_X \qquad \text{........................[3.2]}$$

If the family (B_i) ,with $i = 1,2,...,$is a basis of $\mathfrak{G}$, then the Lie bracket of two elements B_i, B_j of this basis is defined as follows:

$$[B_i ,B_j] = C_{ij}^k B_k, \qquad \text{........................[3.3]}$$

where C_{ij}^k is called the *constant structures in* $\mathfrak{G}\mathfrak{G}_X$. Note that if $\mathrm{M_n}(\mathcal{A}_X)$ is the sheaf of square matrices of *n*-order, with elements in $\mathcal{A}_X$, then the sheaf

$$Gl(n\ ,\ \mathcal{A}_X) = \{A\ \epsilon\ M_n(\mathcal{A}_X)/\ det\ A\ \equiv |a_{ij}|\ \neq 0\} \qquad \text{........................[3.4]}$$

is a Lie group sheaf, where *det A* designs the determinant of $A\ \epsilon\ M_n(\mathcal{A}_X)$.

Let $dT = (\mathcal{A}_X, d_X, \Omega_X)$ be a differential triad. We define the *matrix differential triad* (see[12]) by setting

$$dT_X^M = (Gl(n\ ,\ \mathcal{A}_X)\ ,d_X^M,\ M_n(\Omega_X))\ , \qquad \text{........................[3.5]}$$

where $d_X^M : Gl(n\ ,\ \mathcal{A}_X) \rightarrow M_n(\Omega_X)$ satisfies for any $A\ ,\ B\ \epsilon\ Gl(n\ ,\ \mathcal{A}_U)$ the Leibniz (product) rule

$$d_U^M(A.B) = ad\ (B^{-1})\ .d_U^M\ (A) + d^M\ (B) \qquad \text{........................[3.6]}$$

with $A \equiv (a_{ij})$, $B \equiv (B_{kl})$, $U \subseteq X$ and $i, j, k, l = 1, 2,..., n$. Note that $ad\ (B^{-1})$ designs the *adjoint matrix* of B^{-1} and

$$d_U^M(A) = A^{-1}.\ ad\ (A)\ ,\ \ d_U^M(B^{-1}) = -ad\ (B)\ .\ d^M\ (B) \qquad \text{........................[3.7]}$$

as above

$$[ad(A)\ ,d_U^M(B)\ ] = A\ .d_U^M(B)\ .A^{-1} \qquad \text{........................[3.8]}$$

Where ad_X: $Gl(n\ ,\ \mathcal{A}_X) \rightarrow \mathrm{End}(Gl(n\ ,\ \mathcal{A}_X))$ is the *adjoint representation.*

Remark 3.3

Here d_X^M is *the matrix differential (or derivative) over X.* The matrix notation is justified through the relation

$$Hom_{\mathcal{A}}(\mathcal{A}_X)^n\ ,\mathcal{A}_X)^n) = M_n(\mathcal{A}_X))\ and\ Gl(n\ ,\ \mathcal{A}_X)) = M_n(\dot{\mathcal{A}_X})\ ) \qquad \text{................................[3.9]}$$

within $\mathcal{A}$-isomorphisms, where $\dot{\mathcal{A}}$ designs *the sheaf of invertible elements of* $\mathcal{A}$*.* Note that $M_n(\dot{\mathcal{A}_X})$ and $Gl(n\ ,\ \mathcal{A}_X)$ are Lie group sheaves. Their Lie algebras are denoted respectively by

$\mathfrak{M}_n(\mathcal{A}_X))$ and $\mathfrak{Gl}(n\ ,\mathcal{A}_X))$

Definition 3.4

Consider the following morphism :

$$[,]: \Xi(X) \times \Xi(X) \rightarrow \Xi(X)\ ,\ \ (s, r) \rightarrow [s\ ,\ r\ ] = sr - rs \qquad [3.\ 10]$$

which satisfies the following properties :

(i) $[s\ ,\ r\ ] = -\ [r\ ,\ s]$, anti – symmetry
(ii) $[s\ ,\ [r\ ,\ t\ ]\ ] +\ [s\ ,\ [r\ ,\ t\ ]\ ] +\ [s\ ,\ [r\ ,\ t\ ]]= 0$, Jacobi identity.

We observe that $[\ ,\ ]$ is a *Lie bracket* and $(\ \Xi(X)\ +\ ,\ \cdot\ ,\ [\ ,\ ]\ )$ is a *Lie algebra of vector fields* of *X* .

Consequently, the sets $((\ \Xi(\mathcal{A})\ ,\ +\ ,\ \cdot\ ,\ [\ ,\ ])$ and $((\ \Xi(X)\ (\Omega),\ +\ ,\ \cdot\ ,\ [\ ,\ ])$ are *Lie algebras of vector fields* of $\mathcal{A}$ and Ω, respectively and we set:

$(\,\Xi\,(\mathcal{A}_X, d_X\,,\,\Omega_X\,) \equiv ((\,\Xi\,(\mathcal{A}_X)\,,\,d_X\,,\,(\,\Xi\,(\Omega_X)) \equiv (\mathcal{A}_{(\Xi(X)}, d_{(\Xi(X)}\,,\,\Omega_{(\Xi(X)}),$

and

$\Xi\,(dT) \equiv dT_{\Xi(X)} = (\mathcal{A}_{\Xi(X)}\,, d_{\Xi(X)}\,,\,\Omega_{\Xi(X)})$ ……………………………..[3. 11]

is a Lie algebra differential triad over $(\mathcal{A}\,,\,\Xi(X))$.

Definitions 3.5

The infinitesimal generators of the Lie algebra $\mathfrak{Gl}(n, \mathcal{A}_X)$ of the Lie group $Gl\,(n, \mathcal{A}_X)$ is defined, for any open U in X by :

$$G_{\alpha,U} = \frac{\partial (g,g')^{\beta}}{\partial g'^{\alpha}} \cdot \frac{\partial}{\partial g^{\beta}} = A_{\alpha}^{\beta} \cdot \frac{\partial}{\partial g^{\beta}}$$ ……………………………..[3.12]

where g, g' ϵ $Gl\,(n,\mathcal{A})(U)$ and α, β = 1,…,n .We define the Kirillov form of the Lie algebra $\mathfrak{Gl}(n, \mathcal{A}_X)$) of the Lie group $Gl\,(n, \mathcal{A}_X)$) *as follows:*

$$Kiril_{\alpha\beta}(\gamma) = -\gamma_C C_{\alpha\beta}^{C}$$ ……………………………..[3.13]

where the $C_{\alpha\beta}^{C}$ are the constant structures and the γ_C are the coefficients of an element γ of the dual $\mathfrak{Gl}^*(n, \mathcal{A}_U) = Hom_{\mathcal{A}}\,(\mathfrak{Gl}\,(n, \mathcal{A}_U), \mathcal{A}_U)$ of *the Lie algebra* $\mathfrak{Gl}\,(n, \mathcal{A}_U)$.

Remark 3.6

Note that the *Kirillov* form is skew-symmetric and it is said to be *closed iff* we have :

$$d_U Kiril_{\alpha\beta}(\gamma)_U = 0$$ ………………………………..[3.14]

or more explicitly,

$$\frac{\partial (Kiril_{\alpha\beta}(\gamma))}{\partial_U \gamma_C} = \frac{\partial (-\gamma_C C_{\alpha\beta}^{C})}{\partial_U \gamma_C} = -C_{\alpha\beta}^{C}$$

Which implies that :

$$C_{\alpha\beta}^{C} = -\frac{\partial (Kiril_{\alpha\beta}(\gamma))}{\partial_U \gamma_C}$$ …………………………………….[3.15]

Definitions 3.7

The *Poisson bracket* is defined through the Kirillov form ,for any open U in X and $\mathcal{E}, \mathcal{F} \in C^{\infty}(\mathfrak{Gl}^*(n, \mathcal{A}_U), \mathcal{A}_U)$ as follows *:*

$$\{\mathcal{E}, \mathcal{F}\} = Kiril_{\alpha\beta}\left(\frac{\partial \mathcal{E}}{\partial_U \gamma_\alpha} \cdot \frac{\partial \mathcal{F}}{\partial_U \gamma_\beta}\right) \equiv Kiril_{\alpha\beta} \frac{\partial \mathcal{E}}{\partial_U \gamma_\alpha} \cdot \frac{\partial \mathcal{F}}{\partial_U \gamma_\beta}$$ …………………………………… [3.16]

where $dT_X^{M^*} = (Gl^*(n\,,\,\mathcal{A}_X)\,,\,d_X^{M^*}\,,\,M_n{}^*(\Omega_X))$ ……………………………………[3.17]

represents the dual differential triad of $dT_X^{M} = (Gl(n\,,\,\mathcal{A}_X)\,, d_X^{M},\; M_n(\Omega_X))$ and consequently

$\delta T_X^{M^*} = (\mathfrak{G}\,l\mathfrak{Gl}^*(n\,,\mathcal{A}_X)\,,\,\delta_X^{M^*}\,,\,\mathfrak{M}_n{}^*(\Omega_X))$ ……………………………………[3.18]

is the dual differential triad of $\delta T_X^{M} = (\mathfrak{Gl}\,\mathfrak{G}\,l(n\,,\,\mathcal{A}_X)\,,\delta_X^{M},\; \mathfrak{M}_n(\Omega_X))$.

The derivative of coadjoint action of a group triad on $\delta T_X^{M^*}$ designed by ad_X^*, allows us to define the Kirillov form on $\delta T_X^{M^*}$ as follows :

$\langle ad_{\zeta,X}^*(\theta), \Upsilon\rangle = \langle \theta, [\zeta\,, \Upsilon]\rangle = Kiril_{\alpha\beta}(\theta)\zeta^{\alpha}\Upsilon^{\beta}$, and $\langle ad_{\mathcal{E},X}^*(\mathrm{e}), \aleph\rangle = \langle \mathrm{e}\,, [\,\mathcal{E}\,, \aleph]\rangle = Kiril_{ij}(\mathrm{e})\mathcal{E}^i\aleph^j$.

Remark 3.8

Note that the Poisson bracket is $\mathcal{A}-$ bilinear, skew-symmetric and verifies the Jacobi identity.

4. Illustration

Consider the matrix static group defined as follows:

$$SG_X = \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & t \\ 0 & 1 & 1 \end{pmatrix} : y, t \in \mathbb{R}_X^{\mathcal{A}} \right\}, \quad \text{[4.1]}$$

Where $\mathbb{R}^{\mathcal{A}}$designs the real underlying of the $\mathbb{K}$-algebra sheaf $\mathcal{A}$.

If we set: $h_X = \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & t \\ 0 & 1 & 1 \end{pmatrix}$ = (t,y) and $h'_X = \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & t \\ 0 & 1 & 1 \end{pmatrix}$ = (t',y'), then we have

$$h_X . h'_X = (t,y) . (t',y') = (t + t', y+y') \quad \text{[4.2]}$$

Let $s\mathfrak{G}_X$ be the Lie algebra of the group SG_X whose infinitesimal generators are T and Y. If we use the tensor notation, we can set : ϑ^α= (t,y) and ϑ'^α = (t',y') so that one would write:

$$SG_{\alpha,X} = \frac{\partial (h_X . h'_X)^\beta}{\partial h'_X{}^\alpha} \cdot \frac{\partial}{\partial h_X{}^\beta} = A_\alpha^\beta \cdot \frac{\partial}{\partial h_X{}^\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_t \\ \partial_y \end{pmatrix}, \quad \text{[4.3]}$$

(at the neutral e = (t,y) = (0,0)) ,where $\partial_t = \frac{\partial}{\partial_t}$ and $\partial_y = \frac{\partial}{\partial_y}$.

It follows that :$\binom{T}{Y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \binom{\partial_t}{\partial_y} = \binom{\partial_t}{\partial_y}$. Then , we obtain T = ∂_t and Y = ∂_y which forms a basis of the Lie algebra $s\mathfrak{G}_X$.

The Lie bracket is :

$$[T,Y] = 0 \quad \text{[4.4]}$$

The *Kirilov* form becomes:

$$Kiril_{\alpha\beta}(\gamma) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma \in \mathfrak{Gl}^*(2, \mathbb{R}_X^{\mathcal{A}}) \quad \text{[4.5]}$$

The Poisson bracket is:

$$\{\mathcal{E}, \mathcal{F}\} = Kiril_{\alpha\beta}\left(\frac{\partial \mathcal{E}}{\partial_U \gamma_\alpha} \cdot \frac{\partial \mathcal{F}}{\partial_U \gamma_\beta}\right) = \left(\frac{\partial \mathcal{E}}{\partial \gamma_1}, \frac{\partial \mathcal{F}}{\partial \gamma_2}\right) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{E}}{\partial \gamma_1} \\ \frac{\partial \mathcal{F}}{\partial \gamma_2} \end{pmatrix} = 0 \quad \text{[4.6]}$$

Hence, this Poisson bracket provides Poisson structure associated to the Static group. We construct the dual ($s\mathfrak{G}_X{}^*$,{ , }) of $s\mathfrak{G}_X$ endowed with the Poisson bracket which is the Poisson Static Manifold.

The differential $\delta_X^{M^*}$*:* $\mathcal{G}\, l\mathfrak{Gl}^*(n, \mathbb{R}_X^{\mathcal{A}}) \rightarrow \mathfrak{M}_n{}^*(\Omega_X)$ *will permit us to extend the notion of* Poisson Static Manifold in $\mathfrak{M}_n{}^*(\Omega_X)$.

5. *Conclusion and Future Work*

The main focuses of our investigation in this article were :

- The differential triads that we present as the basic object through which all fundamental notions are constructed;
- The construction of the dual ($s\mathfrak{G}_X{}^*$,{ , }) of $s\mathfrak{G}_X$ endowed with the Poisson bracket which is the Poisson Static Manifold, through the Lie group and Lie algebra theories.

The future work shall consist to treat the following:

- The *Clifford* connection triad algebras ;
- The application of the *Clifford* connection triad algebras in physic.

VI. REFERENCES

[1] Ablamowich, R., G. Sobczyk, Lectures on Clifford (Geometric) Algebras and Applications. Birkhauser, Berlin, 2004.
[2] Boudelliona, M.S., A.Quadrac, Serre's reduction of linear functional systems INRIA report 7214 , to appear in math coumput. Sci. 2010, 5, 169-177.

[3] Cluzean-A. I. Quadrat, A Homological package for factoring and decomposition linear functional systems. Springer, 2009, 179-196.
[4] Cortella, A. Algèbre de Clifford, *Journal of Algebra* 314, p.252-266, 2007.
[5] Goderment, R.. Topologie algébrique et théorie des faisceaux , Harmann, Paris. 2002
[6] Gonzato, M. Les algèbres de Clifford sur R et G , France. 2005.
[7] Herich, H. – G. Stecker ,Category theory , Boston , 1973.
[8] Kashiwara, M. P. SchapiraCategories and sheaves ,Spinger, 2006.
[9] Lam, T.Y. Problem on projective modules, Springer monograph in Mathematics, Springer Verlag. 2006. 47 pages.
[10] M. Barakat- D.Robert , A meta-package for homological algebra , J.AlgebraAppl, 7(2008), 299-317. Homcalg project.
[11] Mallios, A. Geometry of Vector sheaves . An axiomatic approach to Differential Geometric.Examples and Applications, Kluwer Academic Publishers Dordrecht, 1998.
[12] Mallios, A. Geometry and Physics today , Int. J. Theor. Phys. 45 (2006) , 1557-1593.
[13] Mallios, A. Geometry of Vector sheaves . An axiomatic approach to Differential Geometric , Volume I: Vector sheaves, General Theory, Kluwer Academic Publishers Dordrecht, 1998.
[14]Papatrianfilou, M. H. The Category of Differential Triads , Proceedings of, the 4th Panhellenic Conference on Geometry, Bull. Greek Math. Soc. 44 (2000), 129–141.
[15] Ping, HD.Y. Yang , Vanishing Theorems on complete manifolds with weighted Poincaré inequality and Applications, Nagoya Math, J. 2006. 25-37. . 2012.
[16] Quadrat, A. An introduction to constructive algebraic analysis and its applications.7354 july 2010, INRIA, 237 pages.
[17] Quadrat, A. System and structures: an algebraic analysis approach to mathematical system theory. 2010, 185 pages.
[18] Rotman, J.J. An introduction to homological algebraic. Springer; 2nd edition.2009, 87 pages.
[19] Schapira, P. Algebra and Topology, http:www.math.jussien.fr/shapira/ lectnotes, 1/9/2011.

****